LIFTING FORMULAS, MOYAL PRODUCT, AND FEIGIN SPECTRAL SEQUENCE

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Abstract. It is shown, that each Lifting cocycle $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ ([Sh1], [Sh2]) on the Lie algebra Dif_n of polynomial differential operators on an n-dimensional complex vector space is the sum of two cocycles, its even and odd part. We study in more details the first case $n = 1$. It is shown, that any nontrivial linear combination of two 3-cocycles on the Lie algebra Dif_1, arising from the 3-cocycle Ψ_3, is not cohomologous to zero, in a contradiction with the Feigin conjecture [F]. The new conjecture on the cohomology $H^\bullet_{\mathrm{Lie}}(\mathrm{Dif}_1;\mathbb{C})$ is made.

Introduction

The cocycles $\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots$ on the Lie algebra $\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n)$ of finite matrices over polynomial differential operators on an n-dimensional complex vector space, were constructed in the author's works [Sh1], [Sh2] (the lower index denotes the degree of the cocycle). These cocycles are called Lifting cocycles. It was proved in [Sh3] that

$$H^\bullet_{\mathrm{Lie}}(\mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n);\mathbb{C}) = \wedge^\bullet(\Psi_{2n+1}, \Psi_{2n+3}, \Psi_{2n+5}, \dots).$$

The situation is much more complicated for the pull-backs of the Lifting cocycles under the map of the Lie algebras $\mathrm{Dif}_n \hookrightarrow \mathfrak{gl}^{\mathrm{fin}}_\infty(\mathrm{Dif}_n)$, $\mathcal{D} \mapsto E_{11} \otimes \mathcal{D}$. For example, it is known that the first Lifting cocycle Ψ_{2n+1} on the Lie algebra Dif_n is not cohomologous to zero, but the same problem for the higher cocycles $\Psi_{2n+2l+1}$, $l > 0$ is open, even in the simplest case $n = 1$.

In the present paper we prove that there exists the decomposition of each Lifting cocycle on the Lie algebra Dif_n in the sum of two cocycles, its *even* and *odd* part, $\Psi_{2k+1} = \Psi^{\mathrm{even}}_{2k+1} + \Psi^{\mathrm{odd}}_{2k+1}$. To explain this fact, we work with the Moyal star-product on $\mathbb{C}[p,q]$ instead of the usual product in the algebra of differential operators (in the case $n = 1$). In the terms of the algebra Dif_1 it means, that we work with a nonstanard basis in Dif_1, which can be obtained from the usual basis $\{p^iq^j\}$ in $\mathbb{C}[p,q]$ from the Poincaré–Birkhoff–Witt map.

The reason is that in the Moyal product $f * g = f \cdot g + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \dots$ ($f, g \in \mathbb{C}[p,q]$) the bidifferential operators $B_i(f,g)$ are *skew-symmetric* on f and g for *odd* i and are *symmetric* on f and g for *even* i.

In particular,

$$[f,g] = f * g - g * f = 2(\hbar B_1(f,g) + \hbar^3 B_3(f,g) + \hbar^5 B_5(f,g) + \dots)$$

contains only the bidifferential operators B_{2i+1}, $i \geq 0$. This simple trick leads us directly to the decomposition $\Psi_{2k+1} = \Psi^{\mathrm{even}}_{2k+1} + \Psi^{\mathrm{odd}}_{2k+1}$. In the case $n = 1$ we prove that the

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cocycles Ψ_3^{even} and Ψ_3^{odd} are noncohomologous to zero and to each other, and they define a 2-dimensional space in $H^3(\text{Dif}_1, \mathbb{C})$.

Let us note, that the decomposition $\Psi_{2k+1} = \Psi_{2k+1}^{\text{even}} + \Psi_{2k+1}^{\text{odd}}$ exists only for the Lie algebra Dif_n, *not* for the Lie algebra $\mathfrak{gl}_\infty^{\text{fin}}(\text{Dif}_n)$.

In [F] Boris Feigin proposed a program leading to the calculation of the cohomology $H^\bullet_{\text{Lie}}(\text{Dif}_1; \mathbb{C})$. Let us denote by $\mathfrak{gl}(\lambda)$ the Lie algebra of λ-twisted holomorphic differential operators on $\mathbb{C}P^1$, in the case $\lambda \in \mathbb{Z}$ the algebra $\mathfrak{gl}(\lambda)$ is nothing but the algebra of holomorphic differential operators in the bundle $\mathcal{O}(\lambda)$.

The calculation in [F] is based on the (conjectural) direct description of the term E_1 of the Serre-Hochschild spectral sequence corresponded to the embedding $\iota\colon \mathfrak{gl}(\lambda) \hookrightarrow \text{Dif}_1$ of the Lie algebras, for general $\lambda \in \mathbb{C}$. Also it was made the conjecture stating that this spectral sequence degenerates in the term E_2. It follows from these conjectures, that $H^\bullet(\text{Dif}_1; \mathbb{C}) = \wedge^\bullet(\xi_3, \xi_5, \xi_7, \dots) \otimes S^\bullet(c_4, c_6, c_8, \dots)$ (the lower index denotes the degree of the cocycles). The first part (the description of the term E_1) is based on the "main theorem of the invariant theory" for the Lie algebra $\mathfrak{gl}(\lambda)$; particular, it states, that $H^\bullet(\mathfrak{gl}(\lambda); \mathbb{C}) = \wedge^\bullet(\xi_1, \xi_3, \xi_5, \xi_7, \dots)$ (for general λ). It follows from our results that the last statement is not true. Our conjecture is the following:

(i) $H^\bullet_{\text{Lie}}(\mathfrak{gl}(\lambda); \mathbb{C}) = \wedge^\bullet(\xi_1; \xi_3', \xi_3''; \xi_5', \xi_5''; \xi_7'', \xi_7''; \dots)$ (for general $\lambda \in \mathbb{C}$);
(ii) $H^\bullet_{\text{Lie}}(\text{Dif}_1; \mathbb{C}) = \wedge^\bullet(\xi_3', \xi_3''; \xi_5', \xi_5''; \xi_7', \xi_7'', \dots) \otimes S^\bullet(c_4, c_6, c_8, \dots)$.

It would be very interesting to compare (i) with the computation in the additive K-theory:

$$H^\bullet_{\text{Lie}}\left(\mathfrak{gl}_\infty^{\text{fin}}(\mathfrak{gl}(\lambda)); \mathbb{C}\right) = \wedge^\bullet\left(\int_{\mathbb{C}P^1} \Psi_3; \iota^*\Psi_3, \int_{\mathbb{C}P^1} \Psi_5; \iota^*\Psi_5, \int_{\mathbb{C}P^1} \Psi_7; \dots\right)$$

(for general $\lambda \in \mathbb{C}$) made in [Sh3], Sect. 4.3.

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1. Moyal product

Let $\alpha = \sum_{i,j} \alpha^{ij}\partial_i \wedge \partial_j$, $\alpha^{ij} = -\alpha^{ij} \in \mathbb{C}$, be a bivector field on $\mathbb{C}^d$ with constant coefficients; it is clear, that $[\alpha, \alpha] = 0$. There exists an explicit formula for the star-product on $\mathbb{R}^d$, corresponding to the Poisson structure on $\mathbb{R}^d$, defined by the bivector field α; this product is called the Moyal product. The Moyal product is defined as follows:

$$f * g = f \cdot g + \hbar \sum_{i,j} \alpha^{ij}\partial_i(f)\partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \alpha^{ij}\alpha^{kl}\partial_i\partial_k(f) \cdot \partial_j\partial_l(g) + \dots = \tag{1}$$
$$= \sum_{n=0}^\infty \frac{\hbar^n}{n!} \sum_{i_1,\dots,i_n; j_1,\dots,j_n} \prod_{k=1}^n \alpha^{i_k j_k} \left(\prod_{k=1}^n \partial_{i_k}\right)(f) \times \left(\prod_{k=1}^n \partial_{jk}\right)(g).$$

Here the symbols $\cdot$ and $\times$ denotes the usual commutative product on $\mathbb{C}[x_1, \dots, x_d]$, $\partial_i = \dfrac{\partial}{\partial x_i}$ and $f, g \in \mathbb{C}[x_1, \dots, x_d]$.

One can prove that the star-product defined by the last formula actually defines an *associative* product on $\mathbb{C}[x_1, \dots, x_d]$: $(f*g)*h = f*(g*h)$ for any $f, g, h \in \mathbb{C}[x_1, \dots, x_d]$.

Let us suppose that $d = 2n$, and $\alpha = \partial_1 \wedge \partial_{1+n} + \partial_2 \wedge \partial_{2+n} + \dots + \partial_n \wedge \partial_{2n}$.

We will denote by $(\mathbb{C}[p_1, \dots, p_n; q_1, \dots, q_n], *, \hbar)$ the algebra structure on $\mathbb{C}[p_1, \dots, p_n; q_1, \dots, q_n]$ equipped with the star-product (1). It is well-known that for

any $\hbar$ the algebra $(\mathbb{C}[p_1,\dots,p_n;q_1,\dots,q_n],*,\hbar)$ is isomorphic to the algebra $\mathrm{Dif}_n^\hbar$ of polynomial $\hbar$-differential operators on $\mathbb{C}^n$: $\mathrm{Dif}_n^\hbar=\mathbb{C}[\partial_1,\dots,\partial_n;x_1,\dots,x_n]$, $[\partial_i,x_i]=\hbar$. It is obvious that for any $\hbar\neq 0$ $\mathrm{Dif}_n^\hbar\simeq\mathrm{Dif}_n$.

We want to find the explicit formula for the isomorphism of the algebras $C\colon(\mathbb{C}[p_1,\dots,p_n;q_1,\dots,q_n],*,\hbar)\to\mathrm{Dif}_n^\hbar$ such that $C(p_i)=\partial_i$ and $C(q_i)=x_i$ for all $i=1\dots n$. One can prove that such isomorphism C is unique. The problem is quite nontrivial, because for any $f,g\in\mathbb{C}[p_1,\dots,p_n;q_1,\dots,q_n]$ we have

$$f*g=f\cdot g+\sum_{n\geq 1}\frac{\hbar^n}{n!}B_n(f,g) \tag{2}$$

where bidifferential operators B_n are *symmetric* on f and g for *even* n and are *skew-symmetric* on f and g for *odd* n. In particular, the bracket $[f,g]=f*g-g*f$ contains only *odd* degrees in $\hbar$ for any f,g. On the other hand, it is not true for the algebra $\mathrm{Dif}_n^\hbar$. The map C has the following form:

$$f\mapsto\mathcal{O}(f)+\hbar\sum_{i,j}a_{ij}\mathcal{O}\left(\frac{\partial}{\partial p_i}\frac{\partial}{\partial q_j}f\right)+\hbar^2\sum_{i,j,k,l}a_{ijkl}\mathcal{O}\left(\frac{\partial^2}{\partial p_i\partial p_k}\cdot\frac{\partial^2}{\partial q_j\partial q_l}f\right)+\dots \tag{3}$$

where the numbers $a_{ij},a_{ijkl},\dots$ does not depend on f, and

$$\mathcal{O}(p_1^{i_1}\cdot\dots\cdot p_n^{i_n}\cdot q_1^{j_1}\cdot\dots\cdot q_n^{j_n})=x_1^{j_1}\cdot\dots\cdot x_1^{j_n}\cdot\partial_1^{i_1}\cdot\dots\cdot\partial_n^{i_n}.$$

Let us recall the following result from [K], Sect. 8.3.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$, $U(\mathfrak{g})$ be its universal enveloping algebra, $\alpha=\sum c_{ij}^k x_k\dfrac{\partial}{\partial x_i}\wedge\dfrac{\partial}{\partial x_j}$ be the Poisson–Kirillov bivector field on $\mathfrak{g}^*$. Let us denote by $(Sym(\mathfrak{g}),*)$ the star-product on the algebra of polynomial functions given in [K], Sect. 2; this star-product is a generalization of the Moyal product in the case of *linear* functions α^{ij}. It is proved in [K], Sect. 8.3.1 that there exists the unique isomorphism of algebras $I_{alg}\colon U(\mathfrak{g})\to(Sym(\mathfrak{g}),*)$ such that $I_{alg}(\gamma)=\gamma$ for $\gamma\in\mathfrak{g}$.

Also, we denote by I_{PBW} the isomorphism of the vector spaces $I_{PBW}\colon Sym(\mathfrak{g})\to U(\mathfrak{g})$ defined as

$$\gamma_1\cdot\gamma_2\cdot\dots\cdot\gamma_n\mapsto\frac{1}{n!}\sum_{\sigma\in\Sigma_n}\gamma_{\sigma(1)}\cdot\gamma_{\sigma(2)}\cdot\dots\cdot\gamma_{\sigma(n)}$$

For any finite-dimensional Lie algebra $\mathfrak{g}$ there exists a canonical element in $S^k(\mathfrak{g}^*)$ for any $k\in\mathbb{Z}_{\geq 1}$; namely, this is the map $\mathrm{Tr}_k\colon\gamma\mapsto\mathrm{Tr}\left((\mathrm{ad}(\gamma))^k\right)$, $k\geq 1$, $\gamma\in\mathfrak{g}$. We consider elements of $S^k(\mathfrak{g}^*)$ as differential operators with constant coefficients acting on $Sym(\mathfrak{g})$.

The following statement was proved in [K], Sect. 8.3.3.2:

Theorem. *The map $I_{alg}\circ I_{PBW}\colon Sym(\mathfrak{g})\to Sym(\mathfrak{g})$ is equal to* $\exp(\sum_{k\geq 1}c_{2k}\cdot Tr_{2k})$ *where the numbers $c_2,c_4,c_6,\dots$ do not depend on the Lie algebra $\mathfrak{g}$.*

The coefficients $c_2,c_4,c_6,\dots$ can be easily expressed through the Bernoulli numbers.

In our case $\mathrm{Tr}_{2k}\equiv 0$ for any $k\geq 1$, and $C=I_{alg}^{-1}=I_{PBW}$.

2. Cocycle Ψ_3 on the Lie algebra Dif_1

2.1. Let $\mathfrak{A}$ be an associative algebra with trace functional $\mathrm{Tr}\colon\mathfrak{A}\to\mathbb{C}$ (i.e. $\mathrm{Tr}(a\cdot b)=\mathrm{Tr}(b\cdot a)$ for any $a,b\in\mathfrak{A}$), and let D_1,D_2 be two its (exterior) derivations such that:

(i) $\mathrm{Tr}(D_iA)=0$ for $i=1,2$ and any $A\in\mathfrak{A}$;

(ii) $[D_1, D_2] = \mathrm{ad}\, Q$ is an *inner* derivation, $Q \in \mathfrak{A}$.

In these conditions, one can construct ([Sh1], [KLR]) a 3-cocycle $\Psi_3(A_1, A_2, A_3)$ on the Lie algebra $\mathfrak{A}$ (i.e. the Lie algebra obtained from the associative algebra $\mathfrak{A}$ with the bracket $[a, b] = a \cdot b - b \cdot a$).

The 3-cocycle Ψ_3 is defined as follows:

$$\Psi_3(A_1, A_2, A_3) = \operatorname*{Alt}_{A,D} \mathrm{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3) + \operatorname*{Alt}_{A} \mathrm{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3). \tag{4}$$

One can easily check, that Ψ_3 is in fact a cocycle on the Lie algebra $\mathfrak{A}$.

This cocycle is the first nontrivial Lifting cocycle. We refer reader to [Sh1], [Sh2] for the general construction of the Lifting cocycles.

2.2. Let $\mathfrak{A} = (\mathbb{C}[p^{\pm 1}, q^{\pm 1}], *)$ be the associative algebra $\mathbb{C}[p^{\pm 1}, q^{\pm 1}]$ with the $\hbar$-Moyal product (see Sect. 1). we define in this case the trace Tr and the derivations D_1, D_2, which are analogous to the noncommutative residue and the derivations $\mathrm{ad}(\ln x)$ and $\mathrm{ad}(\ln \partial)$, defined in the case of $\mathfrak{A} = \Psi\mathrm{Dif}_1$, where $\Psi\mathrm{Dif}_1$ is the algebra of formal pseudo-differential symbols on $\mathbb{C}^*$ ([KK], [Sh1], [KLR]).

For $f = \sum a_{ij} p^i q^j \in \mathbb{C}[p^{\pm 1}, q^{\pm 1}]$, we set $\mathrm{Tr}(f) = a_{-1,-1}$. It is easy to check that in this way we define a trace functional.

Moreover, we set

$$D_1 f = (\ln p) * f - f * (\ln p),$$
$$D_2 f = (\ln q) * f - f * (\ln q).$$

Let us explain this expression: here $*$ denotes the $\hbar$-Moyal product, and the right-hand sides of both formulas lies in $\mathbb{C}[p^{\pm 1}, q^{\pm 1}]$. It is easy to see that condition (i) from Subsec. 2.1 holds.

Also, one have

(5)
$$[D_1, D_2] = 2\,\mathrm{ad}\left(\hbar p^{-1} q^{-1} + \hbar^3 \cdot \frac{2}{3} p^{-3} q^{-3} + \ldots + \hbar^{2k+1} \cdot \frac{(2k)!}{2k+1} \cdot p^{-(2k+1)} q^{-(2k+1)} + \ldots\right)$$

It is interesting to compare (5) with the analogous calculation in $\Psi\mathrm{Dif}_1([\partial, x] = \hbar)$:

$$[\mathrm{ad}(\ln \partial), \mathrm{ad}(\ln x)] =$$
$$= \mathrm{ad}\left(\hbar x^{-1} \partial^{-1} + \hbar^2 \cdot \frac{1}{2} x^{-2} \partial^{-2} + \hbar^3 \cdot \frac{2}{3} x^{-3} \partial^{-3} + \cdots + \hbar^k \frac{(k-1)!}{k} x^{-k} \partial^{-k} + \ldots\right)$$

2.3. For any two functions $f, g \in \mathbb{C}[p^{\pm 1}, q^{\pm 1}]$ the bracket $f * g - g * f$ (in the sense of the $\hbar$-Moyal product) contains only terms with *odd* degree of $\hbar$:

$$f * g - g * f = 2\left(\hbar \cdot B_1(f, g) + \hbar^3 \cdot B_3(f, g) + \ldots + \hbar^{2k+1} \cdot B_{2k+1}(f, g) + \ldots\right) \tag{6}$$

where

$$f * g = \sum_{i \geq 0} \hbar^i \cdot B_i(f, g),$$

because bidifferential operators B_i are *symmetric* on f, g for *even* i and are *skew-symmetric* on f, g for *odd* i (see Sect. 1).

Let us write:

$$\Psi_3(f_1, f_2, f_3) = \sum_{i \geq 0} \hbar^i \cdot \Psi_3^{(i)}(f_1, f_2, f_3) \tag{7}$$

where $\Psi_3^{(i)}$ do not depend on $\hbar$. Also Ψ_3 is a cocycle. The last fact implies that

$$\Psi_3^{\text{even}}(f_1, f_2, f_3) = \sum_{i\geq 0} \hbar^{2i}\Psi_3^{(2i)}(f_1, f_2, f_3) \tag{8}$$

and

$$\Psi_3^{\text{odd}}(f_1, f_2, f_3) = \sum_{i\geq 0} \hbar^{2i+1}\Psi_3^{(2i+1)}(f_1, f_2, f_3) \tag{9}$$

are cocycles: it follows directly from (6) and the formula for the cochain differential. Therefore, Ψ_3^{even} and Ψ_3^{odd} remain cocycles when $\hbar = 1$.

2.4. We claim that there does not exist any nontrivial linear combination of the cocycles Ψ_3^{even} and Ψ_3^{odd} which is cohomologous to zero.

To prove this, let us note that for any $\lambda \in \mathbb{C}$ the chains

$$c_\lambda = p \wedge (qp + \lambda) \wedge (q^2 p + 2\lambda q) \tag{10}$$

form an one-parametric family of the cycles on the Lie algebra $\mathbb{C}[p^{\pm 1}, q^{\pm 1}]$ with the $\hbar$-Moyal product.

Let us calculate the values of both cocycles Ψ_3^{even} and Ψ_3^{odd} on the cycles c_λ, $\lambda \in \mathbb{C}$.

The only nonzero terms in the first summand of (4) are

$$\mathrm{Tr}(D_1(q^2p + 2\lambda q) * D_2(p) * (qp + \lambda)) \quad \text{and} \tag{11}$$

$$\mathrm{Tr}(D_2(p) * D_1(q^2p + 2\lambda q) * (qp + \lambda)), \tag{12}$$

both with the sign "+".

We have:

$$[\ln p, q^2 p + 2\lambda q]_* = 4\hbar(q + \lambda p^{-1}),$$
$$[\ln q, p]_* = -2\hbar q^{-1},$$

and

$$(11) = -8\hbar^2(\lambda^2 + \lambda\hbar),$$
$$(12) = -8\hbar^2(\lambda^2 - \lambda\hbar).$$

The second summand in (4) is equal to

$$\mathrm{Tr}(Q * (4\hbar^2 pq + 2\lambda\hbar^2 - 4\hbar\lambda^2)). \tag{13}$$

We have:

$$(13) = 2\hbar(2\lambda\hbar^2 - 4\hbar\lambda^2).$$

Finally,

$$\Psi_3^{\text{even}}(c_\lambda) = -16\hbar^2\lambda^2 - 8\hbar^2\lambda^2 = -24\hbar^2\lambda^2,$$
$$\Psi_3^{\text{odd}}(c_\lambda) = 4\lambda\hbar^3.$$

Let $\hbar = 1$, $\lambda_1, \lambda_2 \in \mathbb{C}$. Then the matrix

$$\begin{pmatrix} \Psi_3^{\text{even}}(c_{\lambda_1}) & \Psi_3^{\text{odd}}(c_{\lambda_1}) \\ \Psi_3^{\text{even}}(c_{\lambda_2}) & \Psi_3^{\text{odd}}(c_{\lambda_2}) \end{pmatrix}$$

is equal to

$$A = \begin{pmatrix} -24\lambda_1^2 & 4\lambda_1 \\ -24\lambda_2^2 & 4\lambda_2 \end{pmatrix}.$$

Its determinant is equal to

$$\det A = -4 \cdot 24(\lambda_1^2 \cdot \lambda_2 - \lambda_1 \cdot \lambda_2^2) \tag{14}$$

which is not equal to 0 for some λ_1, λ_2, and therefore the 3-cocycles Ψ_3^{even} and Ψ_3^{odd} are different.

2.5. In the same way one can show that both $(2n+1)$-cocycles $\Psi_{2n+1}^{\mathrm{even}}$ and $\Psi_{2n+1}^{\mathrm{odd}}$ on the Lie algebra $\mathbb{C}[p_1^{\pm 1}, \dots, p_n^{\pm 1}, q_1^{\pm 1}, \dots, q_n^{\pm 1}]$ with the Moyal product are different, where Ψ_{2n+1} is the first Lifting cocycle on this Lie algebra (see [Sh1], [Sh2]).

Conjecture. *The decomposition $\Psi_{2n+2l+1} = \Psi_{2n+2l+1}^{\mathrm{even}} + \Psi_{2n+2l+1}^{\mathrm{odd}}$ of the Lifting cocycle $\Psi_{2n+2l+1}$ on the Lie algebra Dif_n, $l \geq 0$, defines two noncohomologous cocycles, and any their nontrivial linear combination is not cohomologous to zero.*

Let us note, that for $l > 0$ even more simpler fact, that $\Psi_{2n+2l+1}$ itself is not cohomologous to zero is not proved.

3. Lie algebra $\mathfrak{gl}(\lambda)$ and Feigin spectral sequence

Here we consider a consequence of Conjecture 2.5 in the simplest case $n = 1$. According to this Conjecture, one can construct two $(2i+1)$-cocycles $\Psi_{2i+1}^{\mathrm{even}}$ and $\Psi_{2i+1}^{\mathrm{odd}}$ on the Lie algebra Dif_1 for any $i \geq 1$.

We denote, following [F], by $\mathfrak{gl}(\lambda)$ the Lie algebra of holomorphic λ-twisted differential operators on $\mathbb{C}P^1$, $\lambda \in \mathbb{C}$. In the explicit terms,

$$\mathfrak{gl}(\lambda) = \mathrm{Lie}\left(U(\mathfrak{sl}_2)/\Delta - \frac{\lambda(\lambda+2)}{2} \right),$$

where

$$\Delta = e \cdot f + f \cdot e + \frac{h \cdot h}{2} \in U(\mathfrak{sl}_2)$$

is the Casimir element.

For any λ we have an embedding of the Lie algebras $\mathfrak{gl}(\lambda) \hookrightarrow \mathrm{Dif}_1$, and we can consider the Lie algebra $\mathfrak{gl}(\lambda)$ as a subalgebra in the Lie algebra Dif_1. The Feigin spectral sequence is, by definition, the corresponding cohomological Serre–Hochschild spectral sequence, convergating to the cohomology $H^\bullet(\mathrm{Dif}_1; \mathbb{C})$. Its first term

$$E_1^{p,q} = H^q(\mathfrak{gl}(\lambda); \wedge^p(\mathrm{Dif}_1/\mathfrak{gl}(\lambda))^*).$$

The following conjectures were made for general value of $\lambda \in \mathbb{C}$.

The first conjecture made in [F] describes explicitly the algebra $E_1^{\bullet\bullet}$. For any $p \geq 1$ there exist an element c_{2p} in $H^p(\mathfrak{gl}(\lambda); \wedge^p(\mathrm{Dif}_1/\mathfrak{gl}(\lambda))^*)$, and the algebra $E_1^{\bullet\bullet}$ is generated freely by these elements as $H^\bullet(\mathfrak{gl}(\lambda); \mathbb{C})$-module. Furthermore, according to this conjecture,

$$H^\bullet(\mathfrak{gl}(\lambda); \mathbb{C}) = \wedge^\bullet(\xi_1, \xi_3, \xi_5, \xi_7, \dots),$$

and

$$E_1^{\bullet\bullet} = \wedge^\bullet(\xi_1, \xi_3, \xi_5, \xi_7, \dots) \otimes S^\bullet(c_2, c_4, c_6, \dots).$$

The second conjecture states that $d_2(\xi_1) = c_2$, and the spectral sequence degenerates in the term E_2. It follows from these conjectures, that

$$H^\bullet(\mathrm{Dif}_1; \mathbb{C}) = \wedge^\bullet(\xi_3, \xi_5, \xi_7, \dots) \otimes S^\bullet(c_4, c_6, c_8, \dots).$$

In particular, it follows from these computations that $\dim H^3(\mathrm{Dif}_1;\mathbb{C}) = 1$, in a contradiction with the computations of Section 2. Therefore, these conjectures are not true. It seems that the following conjecture is true.

Conjecture. *For general value of $\lambda \in \mathbb{C}$:*

(i) $H^\bullet(\mathfrak{gl}(\lambda);\mathbb{C}) = \wedge^\bullet(\xi_1;\xi_3',\xi_3'';\xi_5',\xi_5'';\xi_7',\xi_7'';\dots)$;

(ii) $E_1^{\bullet\bullet} = \wedge^\bullet(\xi_1;\xi_3',\xi_3'';\xi_5',\xi_5'';\dots)\otimes S^\bullet(c_2,c_4,c_6,\dots)$;

(iii) $d_2(\xi_1) = c_2$, *the spectral sequence degenerates in the term* E_2;

(iv) $H^\bullet(\mathrm{Dif}_1;\mathbb{C}) = \wedge^\bullet(\xi_3',\xi_3'';\xi_5',\xi_5'';\xi_7',\xi_7'';\dots)\otimes S^\bullet(c_4,c_6,c_8,\dots)$.

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